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K-theory and Ramond-Ramond charge

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ABSTRACT: We discuss the relation between the Ramond-Ramond charges of D-branes and the topology of Chan-Paton vector bundles. We show that a topologically nontrivial normal bundle induces RR charge and that the result fits in perfectly with the proposal that D-brane charge is the topology of the Chan-Paton bundle, regarded as an element of K-theory.

KEYWORDS: Branes in String Theory, M- and F-theories and Other Generalizations.

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Contents

1 Introduction

When string theory is compactified on a spacetime $\mathcal{S}$ the low energy effective theory typically contains many $U(1)$ gauge fields. In this paper we will focus on the RR gauge fields arising from KK reduction of the RR gauge fields of ten-dimensional type IIA and IIB supergravity. The vector space of these gauge fields is dual to a space of harmonic forms in $\mathcal{S}$, and hence RR charges take values in $H^*(\mathcal{S};\mathbb{R})$. Quantization of RR charge implies that, in fact, the charge takes values in the $\mathbb{Z}$-module $H^*(\mathcal{S};\mathbb{Z})$.[1]

Perturbative string states all have RR charge zero, but in nonperturbative string theory part of the spectrum is associated with states formed by wrapping D-branes around supersymmetric cycles $\mathcal{W}$ in $\mathcal{S}$ [1, 2]. The data specifying a wrapped D-brane state includes a choice of vector-bundle $E \to \mathcal{W}$, called the Chan-Paton bundle, and a connection on E. In contrast to perturbative states, wrapped D-branes are charged under the RR gauge fields [3]. In this note we will derive a formula for the RR charge of the wrapped D-brane in terms of the topology of the embedded cycle $f : \mathcal{W} \hookrightarrow \mathcal{S}$ and the topology of E. Our main result is

$$Q = \mathrm{ch}\,(f_! E)\,\sqrt{\widehat{A}(\mathcal{TS})}\,, \tag{1.1}$$

where $\mathcal{TS}$ is the tangent bundle to spacetime and $f_!$ is the K-theoretic Gysin map. As we discuss below this strongly suggests that the proper conceptual home for RR charge is in fact the $\mathbb{Z}$-module $K(\mathcal{S})$ rather than $H^*(\mathcal{S};\frac{1}{N}\mathbb{Z})$. The result eq. (1.1) generalizes the result of [4]. The main difference is that in [4] the normal bundles to the branes were assumed to be topologically trivial.

A closely related paper with some overlapping results has recently appeared [5].

[1] Actually, it appears that one should replace $\mathbb{Z} \to \frac{1}{N}\mathbb{Z}$ for some sufficiently large N.

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2 Derivation

The derivation of eq. (1.1) proceeds along the lines of [4], except we will relax the assumption made in [4] that the normal bundles have trivial topology. In general, if $f: \ X \hookrightarrow Y$ is an embedding, we will denote by $\mathcal{N}(X,Y)$ the normal bundle defined by:

$$0 \to TX \xrightarrow{f_*} TY \to \mathcal{N}(X,Y) \to 0\,. \tag{2.1}$$

Let $f: \mathcal{W} \hookrightarrow \mathcal{S}$ be the embedding of the Dp-brane worldvolume into spacetime.[2] The (anomalous) coupling on the worldvolume of N coincident D-branes may be written in the form:

$$I_{\mathcal{W}} = \int_{\mathcal{W}} c \wedge Y(\mathcal{F}, g)\,, \tag{2.2}$$

where $\mathcal{W}$ is the $p+1$-dimensional world-volume of the brane, $c = f^* C$ is the pullback of the total RR potential C, $\mathcal{F} = F - f^* B$ where B is the NS 2-form potential, F is the Hermitian field strength of the $U(N)$ gauge field on the brane, and g is the restriction of the spacetime metric to the brane. Defining $\mathrm{ch}(E) = \mathrm{tr}_N \exp\left(\frac{\mathcal{F}}{2\pi}\right)$, [4] obtained the formula:

$$Y(\mathcal{F}, g) = \mathrm{ch}(E) f^* \sqrt{\widehat{A}(\mathcal{TS})}\,. \tag{2.3}$$

As in [4] we will deduce the anomalous coupling $Y(\mathcal{F}, g)$ from an anomaly inflow argument. Consider two branes intersecting on $\mathcal{I} \equiv \mathcal{W}_1 \cap \mathcal{W}_2$ such that there are chiral fermions on $\mathcal{I}$. To fix ideas, consider two sevenbranes intersecting on a fivebrane.[3] In this situation there are massless chiral fermions moving on $\mathcal{I}$. Let us consider the quantum numbers of these fermions. First, using standard D-brane techniques one finds that if the Chan-Paton bundles on $\mathcal{W}_1, \mathcal{W}_2$ have ranks N_1, N_2, respectively, the fermions are in a hypermultiplet of the gauge group $U(N_1) \times U(N_2)$ in the representation $(N_1, \bar{N}_2) \oplus (\bar{N}_1, N_2)$.

There is another set of fermion quantum numbers associated with the normal bundles $\mathcal{N}_i \equiv \mathcal{N}(\mathcal{I}, \mathcal{W}_i)$ of $\mathcal{I}$. According to [6] the scalar fields in the hypermultiplets are sections of these normal bundles. Let us see what this implies for the quantum numbers of the fermions.

We choose local coordinates at the intersection so that the two sevenbranes are in the 67 and 89 directions. The little group of the $5+1$ theory includes $G = Spin(4)_{2345} \times [Spin(2)_{67} \times Spin(2)_{89}]$. The latter factor is a subgroup of $Spin(4)_{6789}$. We decompose the Lie algebra as: $so(4)_{6789} = su(2)_- \oplus su(2)_+$ where $su(2)_-$ is the $\mathcal{R}$-symmetry of $d = 6, \mathcal{N} = (1,0)$ supersymmetry. The presence of the $\mathcal{I}$ brane breaks the internal symmetry to $Spin(2)_{67} \times Spin(2)_{89}$. Denote the generators of $so(2)_- \oplus so(2)_+$ as $T_- = \frac{1}{2}(T_{67} - T_{89})$ and $T_+ = \frac{1}{2}(T_{67} + T_{89})$, respectively. Globally, $Spin(2)_{67}$ and $Spin(2)_{89}$ are the structure groups of principal $SO(2)$ bundles over $\mathcal{I}$. The normal bundles $\mathcal{N}$ are associated by the vector representation.

[2]We assume embeddings for convenience. The discussion should be generalized to allow immersions.

[3]In the IIB theory, $\mathcal{N}(\mathcal{W}, \mathcal{S})$ is always even. In the IIA theory it is odd. We derive formulae for the charges in IIB and relate them to IIA by T-duality along some circle S^1.

Now consider the string oscillator quantization. The 5+1 worldvolume bosons on the $\mathcal{I}$-brane come from the NS sector. Quantization of the fermion zeromodes $\psi_0^{6,7,8,9}$ gives two complex bosons transforming under G as: $(1,1;T_+ + \frac{1}{2}, T_+ - \frac{1}{2}) + (1,1;T_+ - \frac{1}{2}, T_+ + \frac{1}{2})$. The 5+1 worldvolume fermions on the $\mathcal{I}$-brane come from the Ramond sector. Quantization of the fermions zeromodes $\psi_0^{2,3,4,5}$ gives states transforming under G as $(1,2;T_+,T_+)$. In order for the bosons to be sections of the normal bundle the hypermultiplet must have charge 1/2 under T_+. We conclude that the worldvolume fermions are sections of the spinor bundles associated to $\mathcal{N}_i$.[4]

Now that we have the quantum numbers of the chiral fermions on $\mathcal{I}$ we can compute the anomaly applying the standard descent procedure to:[5]

$$X(\mathcal{I}) = \mathrm{ch}_{N_1}(E_1)\mathrm{ch}_{N_2}(E_2)e^{\frac{1}{2}d_1+\frac{1}{2}d_2}\widehat{A}(\mathcal{T}\mathcal{I})\,. \tag{2.4}$$

where $d_i = c_1(\mathcal{N}_i)$.

We now rewrite eq. (2.4) in a form which makes clear how to cancel the $\mathcal{I}$-brane fermion anomaly via inflow due to the anomalous couplings eq. (2.2) on $\mathcal{W}_i$. When the intersections of $\mathcal{W}_1$ and $\mathcal{W}_2$ are chiral, the brane configurations fill all spacetime dimensions. It follows that

$$0 \to \mathcal{T}\mathcal{I} \to \mathcal{T}\mathcal{W}_1 \oplus \mathcal{T}\mathcal{W}_2 \xrightarrow{\psi} \mathcal{T}\mathcal{S} \to 0 \tag{2.5}$$

is an exact sequence where $\psi(\xi_1 \oplus \xi_2) = \xi_1 - \xi_2$. Recalling that $\widehat{A}$ is a multiplicative characteristic class, $\widehat{A}(E \oplus F) = \widehat{A}(E)\cdot\widehat{A}(F)$, it is easy to see that

$$\widehat{A}(\mathcal{T}\mathcal{I}) = f^*\frac{\widehat{A}(\mathcal{T}\mathcal{W}_1)\cdot\widehat{A}(\mathcal{T}\mathcal{W}_2)}{\widehat{A}(\mathcal{T}\mathcal{S})} \tag{2.6}$$

and hence the anomaly from eq. (2.4) can be cancelled by modifying[6]

$$Y \to Y' = Y e^{\frac{1}{2}d}\frac{\widehat{A}(\mathcal{T}\mathcal{W})}{f^*\widehat{A}(\mathcal{T}\mathcal{S})} \tag{2.7}$$

in eq. (2.3). Here d is a degree two class defining a $Spin^c$ structure on $\mathcal{W}$. It can induce charge shifts of degree two on D-brane worldvolumes along the lines of [6, 8, 4].

Note that this correction is just $\left(\widehat{A}(\mathcal{N}(\mathcal{W},\mathcal{S}))\right)^{-1}$ and hence reflects the contribution from the normal bundle.One might wonder why we should worry about the full sequence $\widehat{A}\,(\mathcal{N})$. The answer is, as remarked in [7], that one should consider families of branes.

[4]This kind of reasoning resolves a small paradox in F-theory. In F-theory compactifications, it can happen that a supersymmetric cycle $\mathcal{W}$, or even spacetime, does not admit a spin structure. As an example, consider F-theory compactification to six dimensions where the base of the elliptically fibered CY 3-fold is $\mathbb{P}^2$ blown up at an even number of points. However, the sevenbranes induce a $Spin^c$ structure allowing the presence of the fermions. Similarly, the fermions of the M-theory fivebrane are valued in the spinor bundle associated to the normal bundle [7].

[5]Here we correct a factor of two in [4]. This is discussed at length in [5].

[6]An alternative proof involves the isomorphisms of the normal bundles $\mathcal{N}(\mathcal{I},\mathcal{W}_1) \cong \mathcal{N}(\mathcal{W}_2,\mathcal{S})|_{\mathcal{I}}$ and as a consequence, $\mathcal{N}(\mathcal{I},\mathcal{S}) \cong \mathcal{N}(\mathcal{W}_1,\mathcal{S})|_{\mathcal{I}} \oplus \mathcal{N}(\mathcal{W}_2,\mathcal{S})|_{\mathcal{I}}$.

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3 Interpretation of the result

We would now like to explore the conceptual meaning of the modification eq. (2.7). We write the modified coupling on the brane in the form

$$I_{\mathcal{W}} = \int_{\mathcal{W}} c \wedge Y'(\mathcal{W}) = \int_{\mathcal{W}} c \wedge \mathrm{ch}(E)\widehat{A}(\mathcal{TW})e^{\frac{1}{2}d} \cdot \frac{1}{f^*\sqrt{\widehat{A}(\mathcal{TS})}} . \tag{3.1}$$

To obtain the D-brane charge, one studies the RR equation of motion and Bianchi identity coming from eq. (3.1). Note that the naive equation of motion:

$$d * F(C) = \delta(\mathcal{W} \hookrightarrow \mathcal{S})Y'(\mathcal{W}) \tag{3.2}$$

is valid only when the normal bundle is trivial. In general, the LHS of eq. (3.2) is defined on $\mathcal{S}$, while the RHS involves only quantities defined on $\mathcal{W}$. The correct equation is:

$$d * F(C) = f_* \left(\mathrm{ch}(E)\widehat{A}(\mathcal{TW})e^{\frac{1}{2}d} \cdot \frac{1}{f^*\sqrt{\widehat{A}(\mathcal{TS})}} \right) . \tag{3.3}$$

Here we have introduced the Gysin map, or push-*forward* f_* acting on *cohomology*.[7] The map f_* is defined using the Poincaré duality map $\mathcal{D}$ twice — first on $\mathcal{W}$, then on $\mathcal{S}$. That is $f_* = \mathcal{D}_{\mathcal{S}}^{-1} f_*^h \mathcal{D}_{\mathcal{W}}$ where on the RHS f_*^h is the natural push-forward on *homology*. Thus:

$$f_* : H^k(\mathcal{W}, \mathbb{Z}) \to H^{k+r}(\mathcal{S}, \mathbb{Z}) , \tag{3.4}$$

where $r = \mathrm{rank}\mathcal{N}(\mathcal{W}, \mathcal{S})$. The new equation of motion for the RR potential eq. (3.3) is consistently defined in the bulk. It is easy to check that the RHS of eq. (3.4) has the right degree.

We can proceed to write the formula for the pairing of the charge vector $Q \in H^*(\mathcal{S})$ with a homology cycle $\Gamma \in H_*(\mathcal{S})$

$$(Q, \Gamma) = \int_{\Gamma} f_* \left(\mathrm{ch}(E)\widehat{A}(\mathcal{TW})e^{\frac{1}{2}d} \cdot \frac{1}{f^*\sqrt{\widehat{A}(\mathcal{TS})}} \right) . \tag{3.5}$$

To avoid cluttering notation we omit the pullback to Γ. To evaluate this integral, we note that for $\phi \in H^*(\mathcal{W}, \mathbb{Z})$ and $\theta \in H^*(\mathcal{S}, \mathbb{Z})$:

$$f_* \left(\phi \wedge f^*\theta\right) = f_*\phi \wedge \theta . \tag{3.6}$$

This follows from the Thom isomorphism:

$$\begin{aligned} \phi \in H^*(\mathcal{W}, \mathbb{Z}) : \quad & f^* f_* \phi = \chi \wedge \phi \\ \theta \in H^*(\mathcal{S}, \mathbb{Z}) : \quad & f_* f^* \theta = \eta \wedge \theta \end{aligned} \tag{3.7}$$

[7] Strictly speaking, we are dealing not with cohomology classes but with "currents" [9]. We need a δ-function representative of the RHS of eq. (3.3). For further discussion see [5].

where χ is the Euler character of the normal bundle and η is the Poincaré dual of the zero section. Thus eq. (3.5) becomes

$$(Q, \Gamma) = \int_\Gamma f_* \left(\mathrm{ch}(E) \widehat{A}(\mathcal{TW}) e^{\frac{1}{2}d} \right) \frac{1}{\sqrt{\widehat{A}(\mathcal{TS})}} . \tag{3.8}$$

The next step involves the use of a version of the Grothendieck-Riemann-Roch theorem for compact oriented differentiable manifolds. We work in the IIB theory so that $\mathcal{N}(\mathcal{W}, \mathcal{S})$ can be considered as a complex vector bundle on $\mathcal{W}$. It inherits an Hermitian structure from g. Under these circumstances we can apply a theorem of Atiyah and Hirzebruch [10, 11, 12] which states that for a certain class $f_! E$ of $K(\mathcal{S})$ (defined below) we have

$$f_* \left(\mathrm{ch}(E) \widehat{A}(\mathcal{TW}) e^{\frac{1}{2}d} \right) = \mathrm{ch}\left(f_! E \right) \widehat{A}(\mathcal{TS}) . \tag{3.9}$$

Here d is an element in $H^2(\mathcal{W}, \mathbb{Z})$ whose reduction mod2 is $w_2(\mathcal{W}) - f^* w_2(\mathcal{S}) = w_2(\mathcal{N})$. In the special case of connected almost complex manifolds, one gets a very nice expression $d = c_1(\mathcal{W}) - f^* c_1(\mathcal{S}) = -c_1(\mathcal{N})$.[8] The map $f_!$ is an isomorphism analogous to the Thom isomorphism for cohomology. Denoting the pull-back operation for bundles with superscript ! and specializing to our case, we can define $f_!$ as

$$f_! E = \pi^! E \otimes \delta(\mathcal{N}) . \tag{3.10}$$

Here $\pi : \ \mathcal{N} \to \mathcal{W}$ is the projection, so $\pi^! E$ can be extended to a bundle on $\mathcal{S}$. $\delta(\mathcal{N})$ is a K-theoretic analogue of the Thom class, defined as follows. First, $\delta(\mathcal{N})$ is an element of relative K theory for bundles on the tubular neighborhood of $\mathcal{W}$ relative to the sphere bundle, again extended to $K(\mathcal{S})$.[9] In particular, $\delta(\mathcal{N})$ is defined as the isomorphism class of the triple:

$$\delta(\mathcal{N}) \cong (-1)^r \left[\Lambda^{\mathrm{even}}_{\mathbb{C}} \pi^* \mathcal{N} \xrightarrow{\sigma(s)} \Lambda^{\mathrm{odd}}_{\mathbb{C}} \pi^* \mathcal{N} \right] , \tag{3.11}$$

where s is a vector in the unit sphere bundle, and, using the Hermitian structure on $\mathcal{N}$ we can define a contraction $\iota(s)$ and hence the isomorphism: $\sigma(s) = s \wedge -\iota(s)$. If $\mathcal{N}$ admits a spin structure this is Clifford multiplication by s. For further details and definitions see [10, 11, 12].

Using eq. (3.9), we finally arrive at our main result: As an element of $H^*(\mathcal{S})$ the RR charge associated to a D-brane wrapping a supersymmetric cycle embedded in spacetime by $f : \mathcal{W} \hookrightarrow \mathcal{S}$ with Chan-Paton bundle $E \to \mathcal{W}$ is given by:

$$Q = \mathrm{ch}\left(f_! E \right) \sqrt{\widehat{A}(\mathcal{TS})} . \tag{3.12}$$

This result has a natural K-theoretic interpretation, first remarked (in the context of the result of [4]) independently by Graeme Segal and Maxim Kontsevich. It is well-known

[8]This can probably also be deduced using the results in [13], where the role of $\frac{1}{2}c_1(\mathcal{S})$ in $\mathcal{N} = 2$ string compactification was investigated.

[9]$f_!$ defines an isomorphism map $f_! : \ K(\mathcal{W}) \to K_{\mathrm{cpt}}(\mathcal{S})$ [11]. A more physical discussion should replace "relative K-theory" by some version of "relative K-theory with rapid decrease in the vertical direction".

that the Chern character is a ring isomorphism from $K(X,\mathbb{Q})$ to the even-dimensional cohomology $H^{\text{even}}(X;\mathbb{Q})$. Both $K(X,\mathbb{Q})$ and $H^{\text{even}}(X;\mathbb{Q})$ have natural bilinear pairings. The pairing on $H^{\text{even}}(X;\mathbb{Q})$ is just given by $(\omega_1,\omega_2)_{DR} = \int_X \omega_1 \wedge \omega_2$, while the pairing on $K(X)$ is given by the index of the Dirac operator: $(E_1,E_2)_K \equiv \text{ind}\not{D}_{E_1\otimes E_2}$. This may be written in terms of the DeRham pairing as:

$$(E_1,E_2)_K = \left(\text{ch}\,(E_1)\sqrt{\widehat{A}(TX)},\ \text{ch}\,(E_2)\sqrt{\widehat{A}(TX)}\right)_{DR} . \tag{3.13}$$

That is, the modified Chern isomorphism

$$E \to \text{ch}(E)\sqrt{\hat{A}(TX)} \tag{3.14}$$

is an *isometry* with respect to the natural bilinear pairings on $K(X)$ and $H^*(X)$. Thus, the most natural statement of the result (3.12) is that the D-brane charge is given by $f_!(E) \in K(\mathcal{S})$.

4 Discussion and conclusion

We conclude by making a few remarks and pointing to a few possible directions for further research. First, we find the identification of RR charge with classes in K-theory philosophically quite natural and harmonious. It is perfectly consistent with the interpretation [14, 15] of Chan-Paton bundles as coherent sheaves, in the complex setting. However, the K-theoretic interpretation is more general since the derivation does not assume any complex structure on $\mathcal{W}$ or $\mathcal{S}$.

Nevertheless, it would be nice to understand the appearance of $f_!(E)$ as defined in eq. (3.10) (as well as the coherent sheaves of [14, 15]) in a more physical language. We hope to return to this in a future publication.

Strictly speaking, our argument for the identification of Q with $f_!(E)$ only applies over $\mathbb{Q}$. However, both $H^*(X;\mathbb{Z})$ and $K(X)$ can have torsion, and the above discussion emphasizes the possibility that D-branes can have torsion charges (e.g. $\mathbb{Z}_N$ charges) and raises the question as to which group has the physically correct torsion classes. There are cases where the torsion in $K(X)$ differs from the torsion in $H^*(X)$. The most elementary example is $X = S^1$, where $\widetilde{KO}(X) = \mathbb{Z}_2$ while the cohomology has no torsion. As an example of a D-brane with nonzero torsion charge one could consider a type I wrapped D1 string whose Chan-Paton bundle is the Mobius bundle. If this twisting has physical consequences (inducing, say, nontrivial Aharonov-Bohm phases in scattering) then the torsion of K-theory is the relevant one. It would be very interesting to study these torsion charges more generally; they would be the D-brane analogues of the the $\mathbb{Z}_N$ charges on black holes and solitons that were explored in [16].

Another, more practical, application of eq. (3.12) might be to anomaly cancellation in various backgrounds in M- and F-theory. It is possible that the modifications of D-brane charges due to normal bundle topology will shed light on some of the difficulties with fivebrane anomalies that have been discussed in [7].

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